# SYSTEMS OF CONTINUOUS AND DISCONTINUOUS SIMPLE GROUPS.

BY DR. LEONARD E. DICKSON.

(Read at the April Meeting of the Society, 1897.)

## § 1.

*Known systems of discontinuous simple groups.*

The following list gives the orders of the systems of discontinuous simple groups which have thus far been determined. Here $p$ = prime, $m$ and $n$ = integers.

(1) $$p$$

(2) $$\tfrac{1}{2}n! \qquad (n>4)$$

(3) $$\frac{(p^{nm}-1)(p^{nm}-p^{n})\cdots(p^{nm}-p^{nm-n})}{d(p^{n}-1)}$$

$(m, n, p) \neq (2, 1, 2)$ or $(2, 1, 3)$; $d$ is the greatest common divisor of $m$ and $p^n-1$.

(4) $$\tfrac{1}{2}(p^{2nm}-1)\,p^{n(2m-1)}(p^{n(2m-2)}-1)\,p^{n(2m-3)}\cdots(p^{2n}-1)\,p^{n}$$
$$(\text{for } p>2)$$
$$(2^{2nm}-1)\,2^{n(2m-1)}\cdots\cdots(2^{2n}-1)\,2^{n} \qquad (m>2).$$

(5) $$(2^{m}-1)\cdot(2^{2m-2}-1)\,2^{2m-2}(2^{2m-4}-1)\,2^{2m-4}\cdots(2^{2}-1)2^{2}$$
$$(m>2).$$

(6) $$(2^{m}+1)\cdot(2^{2m-2}-1)2^{2m-2}\cdots(2^{2}-1)2^{2} \qquad (m>2).$$

The two triply-infinite systems (3) and (4) were obtained* by generalizing to the Galois Field of order $p^n$ the two doubly-infinite systems set up by Jordan.† The system (3) is obtained in the decomposition of the general linear homogeneous group on $m$ indices, in whose substitutions both indices and coefficients are "marks" of the Galois Field of order $p^n$. Concretely, (3) is the group of

*3. L. E. DICKSON. "The analytic representation of substitutions on a power of a prime number of letters, with a discussion of the linear group."—*Annals of Mathematics*, 1897.

4. L. E. DICKSON. A triply-infinite system of simple groups.—*Quarterly Journal of Mathematics*, vol. 29, 1897.

These results were announced in a paper read before the AMERICAN MATHEMATICAL SOCIETY at Buffalo, August 31, 1896.

† Traité des Substitutions, p. 106 and pp. 176, 178.

linear fractional substitutions of determinant unity on $m-1$ indices. The system (4) is obtained in the decomposition of the Abelian group on $2m$ indices; viz, that subgroup of the general linear homogeneous group on $2m$ indices (in which both coefficients and indices are marks of the Galois Field of order $p^n$), every substitution of which if operating simultaneously on two such sets of $2m$ indices

$$\xi_i, \eta_i ; \rho_i, \sigma_i \qquad (i = 1 \cdots m)$$

multiplies the function

$$\sum_{i=1}^{m} (\xi_i \sigma_i - \eta_i \rho_i)$$

by a constant factor.

The systems (3) and (4) have in common the doubly-infinite system (viz, (3) for $m = 2$ and (4) for $m = 1$) set up first by Professor E. H. Moore* and a few months later by Professor W. Burnside† as a generalization of the well known modular group of order

$$\tfrac{1}{2} p (p^2 - 1), \qquad (p > 3).$$

The system (5) is due to Jordan, ibid, p. 205. It is obtained in the decomposition of the First-Hypoabelian group. For the sake of comparison I have changed the form of the expression giving its order, viz:

$$(P_m - 1) 2^{2m-2} (P_{m-1} - 1) 2^{2m-4} \cdots (P_2 - 1) 2^2 \quad (m > 2)$$

where $$P_s = 2^{2s-1} + 2^{s-1}.$$

The system (6) is obtained in the decomposition of the Second-Hypoabelian group. The order of the latter is given incorrectly in Jordan, ibid, p. 207, as equal to the order of the First-Hypoabelian group. The number of solutions of the congruence (42), p. 207, is $P_n$ and not $2^{2n} - P_n$; the reference in § 282 should be to § 259 and not to § 260. I have not seen the simple groups (6) given explicitly anywhere.

Recently‡ I have succeeded in generalizing the system (5) to a doubly infinite system, the expression for the order being rather complicated. The Second-Hypoabelian

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* Abstract in the BULLETIN OF THE MATHEMATICAL SOCIETY, December, 1893; complete in the Mathematical Papers of the Chicago Congress.

† "On a Class of Groups defined by Congruences," *Proceedings of the London Mathematical Society*, vol. 25, pp. 113-139, February, 1894.

‡ "The First-Hypoabelian group generalized," offered April 1st to the *Quarterly Journal of Mathematics*.

group presents serious difficulties to generalization, as the conditions defining it (Jordan, § 277) do not lead to a group at all in the $GF[p^n]$, $n>1$.

## § 2.

*Systems of finite continuous transformation groups which are simple.**

Through the work of Killing†, which has been carried out with more rigor by Cartan,‡ we know that all continuous finite *simple* transformation groups, aside from five isolated ones, belong to one of the four systems set up by Sophus Lie, each of such fundamental importance in geometry and analysis:

(a) The groups with $l(l+2)$ parameters, isomorphic (= gleichzusammengesetzt) with the general projective group of $R_l$ (= space of $l$ dimensions).

($b_1$) The groups with $l(2l+1)$ parameters, isomorphic with the general projective group of a non-degenerate surface of the second order in $R_{2l}$ and hence also with the largest group of conform transformations in $R_{2l-1}$.

($b_2$) The groups with $l(2l-1)$ parameters, isomorphic with the general projective group of a non-degenerate surface of the second order in $R_{2l-1}$. (Here must $l>2$.)

(c) The groups with $l(2l+1)$ parameters, isomorphic with the general projective group of a linear complex in $R_{2l-1}$.

The five isolated simple groups not falling in these four classes contain respectively 14, 52, 78, 133 and 248 parameters and exist as point-transformation groups in respectively 5, 15, 16, 27 and 57 variables (but no fewer); however, as Berührungstransformation groups in fewer variables.

## § 3.

*Elementary deduction of the groups (c) and proof of their simplicity.*

The following elementary proof had its origin in an attempt to carry over into Lie's continuous group theory the

* Professor LIE has determined four classes of infinite transformation groups which are simple; *e. g.*, all transformations of space of $n$ dimensions; again, all Berührungstransformationen.

† WILHELM KILLING, "Die Zusammensetzung der stetigen endlichen Transformationsgruppen, *Math. Ann.*, vols. 31, 33, 34, 36 (particularly vol. 33).

‡ ÉLIE CARTAN, "Ueber die einfachen Transformationsgruppen," *Leipziger Berichte*, pp. 393–420, 1893; "Sur la structure des groupes simples finis et continus," *Comptes Rendus*, vol. 116, p. 784 (17 Apr., 1893) and p. 962; also Thèse, Paris, Nony., 1894.

investigations of Jordan on Abelian groups (whose decomposition led to the system of simple groups (4) ). It is remarkable how much simpler and more elegant the problem becomes for continuous groups. That the simple groups thus obtained are identical with the groups (*c*) is proved below.

Of the transformations of the general linear homogeneous group on $2n$ variables, we consider those which, when operating simultaneously on two independent sets of variables

$$x_i, y_i; \quad \xi_i, \eta_i \quad (i = 1, 2, \cdots n)$$

leave invariant the function

$$\varphi = \sum_{i=1}^{n} (x_i \eta_i - \xi_i y_i).$$

Let $G$ denote the sub-group of the linear homogeneous group thus defined and $S$ an arbitrary transformation of $G$; viz.:

$$x_i' = \sum_{j=1}^{n} (a_j^{(i)} x_j + c_j^{(i)} y_j) \qquad (i = 1 \cdots n)$$

$$y_i' = \sum_{j=1}^{n} (b_j^{(i)} x_j + d_j^{(i)} y_j)$$

The conditions that $S$ shall leave $\varphi$ invariant are:

$$\sum_{\nu=1}^{n} (a_\mu^{(\nu)} d_\mu^{(\nu)} - b_\mu^{(\nu)} c_\mu^{(\nu)}) = 1, \quad \sum_{\nu=1}^{n} (a_\mu^{(\nu)} d_\lambda^{(\nu)} - b_\mu^{(\nu)} c_\lambda^{(\nu)}) = 0,$$

$$\sum_{\nu=1}^{n} (a_\mu^{(\nu)} b_\lambda^{(\nu)} - b_\mu^{(\nu)} a_\lambda^{(\nu)}) = 0, \quad \sum_{\nu=1}^{n} (c_\mu^{(\nu)} d_\lambda^{(\nu)} - d_\mu^{(\nu)} c_\lambda^{(\nu)}) = 0,$$

$$(\mu, \lambda = 1 \cdots n, \mu \neq \lambda),$$

altogether

$$n + n(n-1) + \frac{n(n-1)}{2} + \frac{n(n-1)}{2} = n(2n-1)$$

conditions.

Now $S$ reduces to the identity if

$$a_i^{(i)} = 1, \; d_i^{(i)} = 1, \quad a_j^{(i)} = 0, \; d_j^{(i)} = 0 \quad (i, j = 1 \cdots n, \; i \neq j)$$
$$c_j^{(i)} = 0, \; b_j^{(i)} = 0 \quad (i, j = 1 \cdots n).$$

Hence we reach the general infinitesimal transformation of $G$ by writing

$$a_i^{(i)} = 1 + a_i^{(i)} \delta t \,, \; d_i^{(i)} = 1 + \delta_i^{(i)} \delta t \,, \; a_j^{(i)} = a_j^{(i)} \delta t, \text{ etc.},$$

and subjecting the $\alpha_j^{(i)}$, $\beta_j^{(i)}$, $\gamma_j^{(i)}$, $\delta_j^{(i)}$ to the conditions following from the above $n(2n-1)$; viz,

$$\alpha_\mu^{(\mu)} + \delta_\mu^{(\mu)} = 0\,, \qquad \delta_\lambda^{(\mu)} + \alpha_\mu^{(\lambda)} = 0$$
$$\beta_\lambda^{(\mu)} - \beta_\mu^{(\lambda)} = 0\,, \qquad \gamma_\mu^{(\lambda)} - \gamma_\lambda^{(\mu)} = 0$$
$$(\mu, \lambda = 1 \cdots n,\ \mu \neq \lambda).$$

The most general infinitesimal transformation of the group $G$ is thus

$$\delta x_i \equiv x_i' - x_i = \sum_{j=1}^{n} (\alpha_j^{(i)} x_j + \gamma_j^{(i)} y_j)$$

$$\delta y_i \equiv y_i' - y_i = \sum_{j=1}^{n} (\beta_j^{(i)} x_j - \alpha_i^{(j)} y_j)$$

where $\quad \beta_j^{(i)} = \beta_i^{(j)},\ \gamma_j^{(i)} = \gamma_i^{(j)} \quad (i, j = 1 \cdots n,\ i \neq j).$

Putting in turn each of the $2n^2 + n$ independent constants remaining equal 1 and all the others zero, we reach the $2n^2 + n$ linearly independent transformations of $G$:

$$Q_{ij} \equiv x_j \frac{\partial f}{\partial x_i} - y_i \frac{\partial f}{\partial y_j} \quad (i, j = 1 \cdots n).$$

$$L_i \equiv y_i \frac{\partial f}{\partial x_i}\,; \quad L_i' \equiv x_i \frac{\partial f}{\partial y_i} \quad (i = 1 \cdots n).$$

$$R_{ij} \equiv x_i \frac{\partial f}{\partial y_j} + x_j \frac{\partial f}{\partial y_i}\,; \quad N_{ij} \equiv y_i \frac{\partial f}{\partial x_j} + y_j \frac{\partial f}{\partial x_i} \quad (i, j = 1 \cdots n,\ i \neq j).$$

The notation $Q_{ij}$, $L_i$, etc., corresponds to that of Jordan, p. 174.

We verify the following *Klammerausdruck* relations:

$$(L_i\, Q_{ii}) = 2\, L_i\,;\ (L_i\, Q_{ij}) = 0\,;\ (L_i\, Q_{jk}) = N_{jk}.$$
$$(L_i\, L_i') = -\, Q_{ii}\,;\ (L_i\, R_{ij}) = -\, Q_{ij}\,;\ (L_i\, N_{ij}) = 0.$$
$$(L_i'\, Q_{ii}) = -\,2\, L_i'\,;\ (L_i'\, Q_{ij}) = -\, R_{ij}\,;\ (L_i'\, Q_{jk}) = 0.$$
$$(L_i'\, N_{ij}) = Q_{jj}\,;\ (L_i'\, R_{ij}) = 0.$$
$$(R_{ij}\, Q_{ii}) = -R_{ij}\,;\ (R_{ij}\, Q_{j}) = -\,2\, L_j'\,;\ (R_{ij}\, N_{ij}) = Q_{ii} + Q_{jj}.$$
$$(N_{ij},\ Q_{ii}) = N_{ij}\,;\ (N_{ij}\, Q_{ij}) = 2\, L_i\,;\ (N_{ij}\, Q_{kj}) = N_{ik}.$$

Lemma: If an invariant subgroup $H$ of $G$ contains the single infinitesimal transformation $L_i$ ($i$=fixed), it contains all the transformations of $G$.

If $H$ contains $L_i$, it contains $N_{ij}$ ($j = 1 \cdots n,\ j \neq i$),

since $$(L_i\, Q_{ji}) = N_{ij}.$$

But $$(N_{ij}\, Q_{ji}) = 2L_j.$$

Hence $H$ would contain every $L_i$ and $N_{ij}$; hence also

$$(L_i R_{ij}) = -\,Q_{ij} \quad (j \neq i)\,;\ (L_i'\, L_i) = Q_{ii}$$

$$(Q_{ii} L_i') = 2\,L_i';\quad (Q_{ij}\, L_i') = R_{ij}.$$

Theorem: The group $G$ is simple.

Suppose, indeed, an invariant subgroup $H$ of $G$ exists. Let it contain the infinitesimal transformation (not the identity),

$$T = \sum_{ij}^{1\ldots n} q_{ij}\, Q_{ij} + \sum_{i=1}^{n} l_i'\, L_i' + \sum_{i=1}^{n} l_i L_i + \sum_{i,j(i\neq j)}^{1\ldots n} (r_{ij} R_{ij} + n_{ij} N_{ij}),$$

where, since $R_{ij} = R_{ji}$, $N_{ij} = N_{ji}$, we may take

$$r_{ij} = r_{ji},\ n_{ij} = n_{ji}.$$

Then must $H$ contain

$$(L_k T) = 2\,q_{kk} L_k + \sum_{i=1\ldots n}{}' q_{ik} N_{ik} - l_k'\, Q_{kk} - \sum_{i=1\ldots n}{}' r_{ik}\, Q_{ki},$$

$k$ being fixed and $\sum'_i$ indicating that $i \neq k$.

Similarly must $H$ contain

$$(L_k' T) = -\,2\,q_{kk} L_k' - \sum_i{}' q_{ki} R_{ki} + l_k\, Q_{kk} + \sum_i{}' n_{ik}\, Q_{ik}.$$

Hence (for $s \neq k$) $H$ contains

$$(L_s (L_k T)) = -2\,l_k' L_s;\ (L_k' (L_k' T)) = -2\,l_k L_k';$$

$$(L_s (L_k T)) = -r_{sk} N_{sk};\ (L_s' (L_k' T)) = -n_{sk} R_{sk};$$

$$(L_s' (L_k T)) = q_{sk}\, Q_{ks}$$

But $(Q_{sk} N_{sk}) = -2\,L_s;\ (N_{sk}(L_s R_{sk})) = -(N_{sk}\, Q_{sk}) = -2\,L_s.$

Since not every $l_k'$, $l_k$, $r_{sk}$, $n_{sk}$, $q_{sk}$, $q_{kk}$ is zero, $T$ not being the identity, we find that $H$ contains certainly one $L_i$ and by the lemma the whole of $G$.

It remains to prove that the simple group $G$ in $2n$ variables is the homogeneous form of the largest projective group in space of $2n - 1$ dimensions (with the coördinates $z, x_1, \cdots x_{n-1}, y_1, \cdots y_{n-1}$) leaving invariant the linear complex defined by the Pfaff's equation:

$$dz + \sum_{i=1}^{n-1} (x_i\, dy_i - y_i\, dx_i) = 0.$$

The $n(2n+1)$ infinitesimal transformations of the latter group are,* if $U \equiv \sum_{\nu=1}^{n-1}(x_\nu p_\nu + y_\nu q_\nu) + zr$:

$$p_i - y_i r,\ q_i + x_i r,\ r,\ zr + U,\ x_i q_k + x_k q_i,$$
$$x_i p_k - y_k q_i,\ y_i p_k + y_k p_i,\ z p_i - y_i U,\ z q_i + x_i U,\ zU.$$
$$(i, k = 1 \cdots n-1)$$

If we introduce† for the variables $x_i$, $y_i$, $z$ respectively $x_i/x_n$, $y_i/x_n$, $y_n/x_n$, we reach the homogeneous group‡ (where the new transformations are written in the same order as the old above):

$$Q_{ik},\ R_{ni},\ L_n',\ -Q_{nn},\ R_{ik} \text{ and } 2L_i',$$
$$Q_{ki},\ N_{ik} \text{ and } 2L_i,\ N_{in},\ -Q_{ni},\ -L_n$$
$$(i, k = 1 \cdots n-1;\ \ i \neq k \text{ in } N_{ik} \text{ and } R_{ik}).$$

This group is evidently identical with our group $G$. By the same transformation the above Pfaff's equation takes the homogeneous form (aside from the factor $1/x_n^2$):

$$\sum_{i=1}^{n}(x_i\,dy_i - y_i\,dx_i) = 0.$$

## § 4.

*Semi-simple linear homogeneous groups whose defining function§ is the sum of n determinants of order $q > 2$.*

Of the transformations of the general linear homogeneous group in $qn$ variables, we consider those which, when operating simultaneously on $q$ independent sets of $n$ $q$ variables, the $j^{\text{th}}$ set of which may be exhibited thus

$$x_{i1}^{(j)}, x_{i2}^{(j)}, \cdots x_{iq}^{(j)}, \qquad (i = 1 \cdots n),$$

leave invariant the function

$$\varphi \equiv \sum_{i=1}^{n} D_i,$$

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* LIE, Theorie der Transformationsgruppen, vol. II., p. 522.

† The formulæ for this change of variables are given in LIE, ibid, vol. I., p. 579.

‡ For $n = 2$, *i. e.*, for 4 variables, this homogeneous form is given in LIE, ibid, vol. II., p. 450.

§ Discontinuous linear groups with the same defining function are mentioned (but not studied) by Jordan, l. c., p. 219. The result reached in this paragraph, where $q > 2$, is wholly different from that of § 3, where $q = 2$.

where $D_i$ denotes the determinant

$$D_i \equiv \begin{vmatrix} x_{i1}^{(1)}, x_{i2}^{(1)}, \cdots x_{iq}^{(1)} \\ x_{i1}^{(2)}, x_{i2}^{(2)}, \cdots x_{iq}^{(2)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_{i1}^{(q)}, x_{i2}^{(q)}, \cdots x_{iq}^{(q)} \end{vmatrix}$$

Denote by $G$ the groups of transformations thus defined and let $S$ be an arbitrary transformation of $G$; viz,

$$x_{ik}' = \sum_{j=1}^{n} (a_{j1}^{ik} x_{j1} + a_{j2}^{ik} x_{j2} + \cdots + a_{jq}^{ik} x_{jq})$$
$$(i = 1 \cdots n; \ k = 1 \cdots q).$$

The conditions that $S$ shall leave $\varphi$ invariant are :

$$(1) \qquad \sum_{i=1}^{n} \begin{vmatrix} a_{j1}^{i1}, a_{j2}^{i1}, \cdots a_{jq}^{i1} \\ a_{j1}^{i2}, a_{j2}^{i2}, \cdots a_{jq}^{i2} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{j1}^{iq}, a_{j2}^{iq}, \cdots a_{jq}^{iq} \end{vmatrix} = 1 \qquad (j=1, 2, \cdots n).$$

$$(2) \qquad \sum_{i=1}^{n} \begin{vmatrix} a_{j_1 k_1}^{i1} & \cdots\cdots & a_{j_1 k_q}^{i1} \\ \vdots & & \vdots \\ a_{j_q k_1}^{iq} & \cdots\cdots & a_{j_q k_q}^{iq} \end{vmatrix} = 0$$

for $j_s = 1 \cdots n$; $k_s = 1 \cdots q$, provided not all the $j_s$'s are equal and no pair $(j_\lambda, k_\lambda) = (j_\mu, k_\mu)$.

The general infinitesimal transformation of $G$ is found by substituting in $S$

$$a_{ik}^{ik} = 1 + a_{ik}^{ik} \cdot \delta t, \ a_{rs}^{ik} = a_{rs}^{ik} \cdot \delta t \begin{pmatrix} i, r = 1 \cdots n; \ k, s = 1 \cdots q \\ (r, s) \neq (i, k) \end{pmatrix}$$

The first set of conditions (1) on the $a$'s give at once

$$a_{j1}^{j1} + a_{j2}^{j2} + \cdots + a_{jq}^{jq} = 0 \qquad (j = 1 \cdots n).$$

To prove that the second set (2) gives simply (for $q > 2$)

$$a_{j_1 m}^{jl} = 0 \ (l, m = 1 \cdots q; \ j, j_1 = 1 \cdots n, \ j_1 \neq j),$$

consider first the conditions in which $j_2 = j_3 = \cdots = j_q = j$, say, and thus $j_1 \neq j$; viz,

$$\sum_{i=1}^{n} \begin{vmatrix} a_{j_1 k_1}^{i1}, a_{jk_2}^{i1}, a_{jk_3}^{i1}, \cdots a_{jk_q}^{i1} \\ \vdots \\ a_{j_1 k_1}^{iq}, a_{jk_2}^{iq}, a_{jk_3}^{iq}, \cdots a_{jk_q}^{iq} \end{vmatrix} = 0$$

For those determinants given by $i \neq j$, the elements of the last $q - 1$ columns all have the factor $\delta t$ and hence

the determinants have the factor $\delta t^{q-1}$ and hence the factor $\delta t^2$. For the determinant $i=j$, the elements of the first column have the factor $\delta t$; likewise those of the $l^{\text{th}}$ row, if $l$ be the integer $\leqq q$ which is lacking among the numbers $k_2, k_3, \cdots k_q$, all different. The minor of $a^{il}_{j_1 k_1}$ has the term

$$\pm a^{il}_{j k_2} a^{l k_3}_{j k_3} \cdots a^{l k_q}_{j k_q} = \pm (1 + a^{l k_2}_{j k_2} \cdot \delta t) \cdots (1 + a^{l k_q}_{j k_q} \cdot \delta t)$$

and but one such. Hence on expanding the determinant according to the elements of the first column, we get the term $\pm a^{il}_{j_1 k_1} \cdot \delta t$ together with terms of higher order in $\delta t$.

Hence $a^{il}_{j_1 k_1} = 0$.

The remaining ones of the conditions (2) are then satisfied identically. For unless $i = j_s$, the elements of the $s^{\text{th}}$ column

$$a^{il}_{j_s k_s} = a^{il}_{j_s k_s} \cdot \delta t = 0 \quad (l = 1 \cdots q).$$

But $i$ can not simultaneously equal $j_1, j_2, \cdots j_q$ for the conditions (2); hence the elements of at least one column of every determinant are all zero.

The group $G$ thus contains $n(q^2-1)$ linearly independent infinitesimal transformations,

$$\delta x_{ik} \equiv x_{ik}' - x_{ik} = a^{ik}_{i1} x_{i1} + a^{ik}_{i2} x_{i2} + \cdots + a^{ik}_{iq} x_{iq}$$

where $$\sum_{l=1}^{q} a^{il}_{ik} = 0 \qquad (i = 1 \cdots n).$$

The finite equations of the transformations of $G$ are then evidently

$$x_{ik}' = a^{ik}_{i1} x_{i1} + \cdots + a^{ik}_{iq} x_{iq} \qquad (i = 1 \cdots n;\ k = 1 \cdots q),$$

where the determinants

$$| a^{ik}_{is} | = 1 \qquad (i = 1 \cdots n).$$

Thus the group $G$ breaks up into $n$ sub-groups, those transformations in which $i$ has a fixed value forming the linear homogeneous group, leaving the determinant $D_i$ invariant, *i. e.*, the special linear homogeneous group in $q$ variables. Since $G$ breaks up into $n$ invariant *simple* sub-groups, and since all the transformations of any one are commutative with all of any other one, $G$ is a so-called half-simple (halbeinfach) group.

LEIPZIG, *February* 6, 1897.

[The contents of § 3 and § 4 were presented February 19th before the Groups-Seminar of Professor Lie, who stated that the interesting result of § 4 was new and not what one would have expected.]

# ON THE NUMBER OF ROOTS OF THE HYPERGEOMETRIC SERIES BETWEEN ZERO AND ONE.

BY MR. M. B. PORTER.

(Read at the March Meeting of the Society, 1897.)

In 1890 Klein* published his solution of the problem of enumerating the roots of the Hypergeometric Series between 0 and 1.

His method depending on the conformal property of the Schwarzian $s$-function, finally turns on a discussion of the shape of the circular triangles on which the $x$-halfplane is mapped.

Solutions by Hurwitz† and Gegenbauer‡ appeared soon after, both depending on the determination of a chain of contiguous hypergeometric functions which could be employed as a set of Sturmian functions.

Klein's method, while it makes use only of the differential equation and yields the desired result in an exceedingly neat form, does not lead to this result so directly or naturally as certain methods of Sturm (Tom. 1, *Liouville's Journal*).

It is the object of this paper to apply two well known theorems of the above mentioned memoir of Sturm to the solution of the problem in hand.

The theorems referred to are:

A. Let $x_1$ and $x_2$ be two regular singular points of the differential equation

$$\frac{d^2\overline{y}}{dx^2} = \varphi(x, a)\,\overline{y} \qquad (1)$$

If there be no singular point between $x_1$ and $x_2$ and all the magnitudes involved be supposed real, the real roots of $\overline{y}_1$ between $x_1$ and $x_2$, $\overline{y}_1$ being the solution corresponding to the larger exponent of $x_1$, will move toward the point $x_1$, if, for all values of $x$ between $x_1$ and $x_2$, $\varphi(x, a)$ decrease with the decrease (increase) of $a$; *i. e.*, $y_1$ is gaining or *at most* not losing roots between $x_1$ and $x_2$.

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* *Math. Ann.*, vol. 37.

† *Math. Ann.*, vol. 38.

‡ *Wiener Sitzungsberichte*, vol. $100^{2a}$.

B. In the interval $x_1\ x_2$, between any two consecutive real roots of a solution of (1) lies one and but one root of a linearly independent solution. A more general form of this theorem is the following: If $\bar{y}_1$ denote the solution corresponding to the larger exponent of $x_1$ and $x_2$ the root in the interval $x_1\ x_2$ nearest $x_1$, any solution linearly independent of $\bar{y}_1$ will have one root between each pair of consecutive roots of $\bar{y}_1$ and one root between $x_1$ and $x_2$.*

The singular points of the hypergeometric equation:

$$\frac{d^2y}{dx^2}+\frac{\gamma-(\alpha+\beta+1)x}{x(1-x)}\frac{dy}{dx}-\frac{\alpha\beta}{x(1-x)}y=0$$

| | | | |
|---|---|---|---|
| are | 0 | $\infty$ | 1 |
| the exponents, | 0 | $\alpha$ | 0 |
| | $1-\gamma$ | $\beta$ | $\gamma-\alpha-\beta$ |

the exponent-differences being

$$\lambda=\gamma-1,\qquad \mu=\alpha-\beta,\qquad \nu=\alpha+\beta-\gamma.$$

The hypergeometric equation is reduced to the binomial form,

$$\frac{d^2\bar{y}}{dx^2}=\varphi(\lambda,\mu,\nu;\ x)\bar{y}$$

$$=-\tfrac{1}{4}\frac{x^2(1-\mu^2)+x(\mu^2-\nu^2+\lambda^2-1)+1-\lambda^2}{x^2(1-x)^2}\bar{y}$$

by the change of dependent variable

$$y=\bar{y}\,x^{-\frac{1+\lambda}{2}}(1-x)^{-\frac{1+\nu}{2}}$$

The method to be explained serves primarily, as does the method used by Klein, to determine the number $X$ of real roots of the solution corresponding to the larger exponent of $x=0$ in the interval 0–1. The word root will always be used to denote a real root lying between 0 and 1. In the case where $\lambda<0$ the determination of the number of roots of the hypergeometric series which is then the solution corresponding to the lesser exponent of $x=0$ is effected by means of B. The treatment of the so-called exceptional cases is that of Klein but in the method here employed appears as a special case of the general discussion.

* These slightly generalized forms of Sturm's theorems can be deduced from the analysis given by Sturm, for other proofs see Professor Bôcher's paper in the March BULLETIN.

Since $$F(\alpha, \beta, \gamma, x) = F(\beta, \alpha, \gamma, x)$$

there is no loss of generality in assuming

$$\mu = \alpha - \beta \geq 0.$$

Consequently

$$\frac{\partial \varphi(\lambda, \mu, \nu, x)}{\partial \mu} = -\tfrac{1}{2}\frac{\mu}{x(1-x)} < 0, 0 < x < 1.$$

Further, since

$$F(\alpha, \beta, \gamma, x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\beta, \gamma-\alpha, \gamma, x) \quad (2)$$

$$F(\gamma-\beta, \gamma-\alpha, \gamma, x)$$

has the same roots as $F(\alpha, \beta, \gamma, x)$

and as it satisfies a differential equation with the exponent differences $\lambda, \mu, -\nu$; we may suppose $\nu \geq, 0$.

By A., the solution $\bar{y}$, $(y,)$ corresponding to the larger exponent of $x = 0$ is gaining roots as $\mu$ increases. Two cases present themselves:—I. $\lambda > 0$, II. $\lambda < 0$.

## I.

$\lambda > 0$. Here 0 is the larger exponent and the corresponding solution is

$$y = F(\alpha, \beta, \gamma, x) = F\left(\frac{1+\lambda+\mu+\nu}{2}, \frac{1+\lambda-\mu+\nu}{2}, 1+\lambda, x\right)$$

$$= a\,F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)$$

$$+ b(1-x)^{-\nu} F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1, 1-x)$$

where $$a = \frac{\Gamma(1+\lambda)\,\Gamma(-\nu)}{\Gamma\left(\frac{1+\lambda-\mu-\nu}{2}\right)\Gamma\left(\frac{1+\lambda+\mu-\nu}{2}\right)}$$

$$b = \frac{\Gamma(1+\lambda)\,\Gamma(2+\nu)}{\Gamma\left(\frac{1+\lambda+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\lambda-\mu+\nu}{2}\right)}$$

Since $\nu > 0$, it is obvious from the series itself that $y$ can have no positive roots unless $\beta = (1+\lambda-\mu+\nu)/2 < 0$. If then $\mu$, starting with a value a little less than $1+\lambda+\nu$, increase, $\beta$ will decrease. For $\beta = 0, -1, -2$, etc., we get a series of polynomials of degree 0, 1, 2, etc. Whenever $\beta$

passes through zero or a negative integer, $b$ changes sign, $\Gamma(\beta)$ becoming infinite. Since $\nu > 0$, the sign of $y$ for values of $x$ a little less than 1, depends only on $b$ and therefore only on $\Gamma(\beta)$. When $\beta = 0$, $y = 1$, which has no vanishing point, however, immediately *after* $\beta$ decreases through 0, $b$ changes sign and consequently $y_{x=1}$ changes sign; so that as $y = 0$ is gaining roots as $\mu$ increases, an odd number of roots must have been gained immediately after $\beta$ passed through 0,—a sudden dropping of the whole curve $y = F(\alpha, \beta, \gamma, x)$ below the x-axis is impossible since $y_{x=0} = 1$. There could have been a gain of but *one* root when $\beta$ decreased through 0, for as $\beta$ keeps on decreasing no roots are lost and when $\beta = -1$, $y = F(\alpha, -1, \gamma, x) = 1 - \frac{\alpha}{\gamma} x$ which has but one root.

In the same way immediately after $\beta$ decreases through $-1$, $b$ changes sign and one root is gained for when $\beta = -2$

$$y = F(\alpha, -2, \gamma, x) = 1 - = \frac{2\alpha}{\gamma} x + \frac{2\alpha(\alpha+1)}{2\gamma(\gamma+1)} x^2$$

which has not more than two roots. $X$, the number of roots, is, therefore, equal to the number of changes of sign of $b$ as $\beta$ decreases from a small positive value to the required negative value, *i. e.*

$$X = E(1-\beta) = E\left(\frac{\mu - \lambda - \nu + 1}{2}\right)$$

which is the number of roots of

$F(\alpha, \beta, \gamma, x) = 0$ when $\lambda > 0, \nu > 0$.

I follow Klein in using $E(n)$ to denote the largest positive integer less than $n$, so that $E(n) = 0$ if $n \gtrless 1$. When

$\nu < 0$, by 2. $X = E(1 - (\gamma - \alpha))$

$$= E\left(\frac{\mu - \lambda + \nu + 1}{2}\right)$$

## II.

When $\lambda < 0$, $1 - \gamma$ is the larger exponent of 0 and the corresponding solution is

$$y = x^{-\lambda} F\left(\frac{1-\lambda+\mu+\nu}{2}, \frac{1-\lambda-\mu+\nu}{2}, 1-\lambda, x\right)$$

$$= x^{-\lambda} F(\alpha', \beta', \gamma', x)$$

The hypergeometric series $F(\alpha', \beta', \gamma', x)$ satisfies a differential equation whose exponent differences are $-\lambda, \mu, \nu$, therefore, by I. it has $X = E(1-\beta')$ roots between 0 and 1 if $\nu > 0$ or if $\nu < 0$ has $X = E\big(1-(\gamma'-\alpha')\big)$ roots.

All four results can be written in one formula

$$X = E\left(\frac{\mu - |\lambda| - |\nu| + 1}{2}\right).$$

In case II. we have determined the number of roots of a solution linearly independent of the hypergeometric series. By B. $F(\alpha, \beta, \gamma, x)$ has when $\lambda < 0$ $N = X$ or $X+1$ roots, the even or odd value of $N$ being chosen according as $y_{x=1}$ is positive or negative. There are thus two cases.

1°. $\nu > 0$. Here, according as $b$ is $>$ or $<0$, *i. e.*, according as

$$\Gamma(1+\lambda)\,\Gamma\left(\frac{1+\lambda+\mu+\nu}{2}\right)\;\Gamma\left(\frac{1+\lambda-\mu+\nu}{2}\right) > \text{or} < 0$$

the sign of $y_{x=1}$ is $>$ or $<0$.

2°. $\nu < 0$. Here according as $a$ is $>$ or $<0$, *i. e.*, according as

$$\Gamma(1+\lambda)\,\Gamma\left(\frac{1+\lambda-\mu-\nu}{2}\right)\;\Gamma\left(\frac{1+\lambda+\mu-\nu}{2}\right) > \text{or} < 0$$

$y_{x=1}$, is $>$ or $<0$.

If in 1° $b=0$, *i. e.*, if $(1+\lambda+\mu+\nu)/2$ or $(1+\lambda-\mu+\nu)/2$ is zero or a negative integer, the hypergeometric series ceases to be linearly independent of $F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)$ which when $\nu > 0$ is the solution corresponding to the larger exponent of $x=1$; so that by B. in this case the hypergeometric series has $X$ roots. For the same reason in 2°, when $a=0$, *i. e.*, when $(1+\lambda-\mu-\nu)/2$ or $(1+\lambda+\mu-\nu)/2$ is zero or a negative integer the hypergeometric series has $X$ roots.

HARVARD UNIVERSITY,
*March* 10, 1897.

# ON MODULAR EQUATIONS.

BY PROFESSOR JAMES PIERPONT.

Read at the March Meeting of the Society, 1897.)

The theory of the equations of transformation has been put in an entirely new light and very essentially improved by H. Weber's paper "Zur Theorie der Elliptischen Functionen.* His starting point is the solution of the equation for the division of the periods making a systematic use of the Galoisian theory of equations. From this standpoint it is not the modular equation

$$M(v, u) = 0$$

with coefficients rational in $u = \sqrt[4]{k}$ that we are led to consider but the equation

$$T(y, x) = 0$$

whose coefficients are rational in $x = k^2$ and whose roots are the $n + 1$ values of

$$\prod_{p=1}^{m} \frac{\mathrm{cn}}{\mathrm{dr}} \left( p \frac{4\lambda K + 4\mu i K'}{n} \right) \qquad \begin{matrix} \lambda, \mu = 0, 1 \cdots n-1 \\ \lambda = \mu = 0 \text{ excluded} \end{matrix}$$

Here for simplicity we take $n = 2m + 1$, an odd prime.

Let us see how the coefficients of this equation can be calculated. As the roots of $T$ differ from those of $M$ only by the factor $u^{-n}$, the $T$ equations could be derived from the modular equations $M = 0$ on setting $v = u^n y$. But the methods given to compute the modular equations compel us—as far as I have been able to consult the literature—to pass from our domain of rationality $R(x)$ to that of $R(u)$, and this from our standpoint is certainly objectionable unless necessary. To show that this is not so is the *first object* of the present paper. Again, the methods given to calculate $M = 0$ are made—as far as I know—to depend upon the transformation theory of Hermite's function $u = \varphi(\tau)$. I propose to show as *second object* that we can calculate our $T$ equation without leaving the $\vartheta$'s. A considerable simplification is thus obtained. But, having simplified the calculation of $T$ so far, I have been tempted to go one step farther and show how we may arrive at Weber's equations

*Acta Mathematica*, vol. 6, p. 329. Also his book Elliptische Functionen und Algebraische Zahlen, Braunschweig, 1891.

of transformation without the Galoisian Theory; this is my *third object*. In this way a tolerably simple and compact theory for calculating the $T$ equation is given.

1.

We show first how we may arrive at Weber's Equations. The multiplication theorem for

$$\mathrm{sn}u,\ \mathrm{cn}u,\ \mathrm{dn}u$$

tells us three things that are of importance here:
1° the $n^2-1$ quantities

$$x_{\lambda,\mu} = \mathrm{sn}\,\frac{\lambda 4K + \mu 4iK'}{n} = \mathrm{sn}w_{\lambda,\mu}$$

are roots of an equation

$$P(x;\varkappa) = 0$$

whose coefficients are rational in $\varkappa$,
2° the quantities

$$\mathrm{cn}w_{\lambda,\mu},\quad \mathrm{dn}w_{\lambda,\mu}$$

are rational* in $x^2_{\lambda,\mu}$

3° $$x_{g\lambda,\,g\mu} = \theta_g\,(x_{\lambda,\mu})$$

where $\theta_g$ is a rational function independent of $\lambda, \mu$.

This last remark shows us that the roots of $P = 0$ can be arranged in the array†

(A)

$$\begin{array}{llll} x_{1,0} & \theta x_{1,0} & \theta^2 x_{1,0} & \cdots\cdots \theta^{n-2} x_{1,0} \\ x_{0,1} & \theta x_{0,1} & \theta^2 x_{0,1} & \cdots\cdots \theta^{n-2} x_{0,1} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ x_{n-1,1} & \theta x_{n-1,1} & \theta^2 x_{n-1,1} & \cdots\cdots \theta^{n-2} x_{n-1,1} \end{array}$$

For let $g$ be a primitive congruence root of $n$, then 3° shows that if

$$x_{g\lambda,\,g\mu} = \theta x_{\lambda,\mu}$$

then

$$x_{g^2\lambda,\,g^2\mu} = \theta x_{g\lambda,\,g\mu} = \theta\cdot\theta x_{\lambda,\mu} = \theta^2 x_{\lambda,\mu} \text{ etc.}$$

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*We take from now on $E(\kappa)$ for domain of rationality. For a simple explanation of this term see my paper in the BULLETIN for December, 1895, p. 81.

†*Cf.* ABEL. Oeuvres, vol. I., p. 478. Sur une classe particulière d'équations, etc.

Let now $t_\infty$ be a rational symmetric function of the roots of the first row of $A$

$$t_\infty = f(x_{1,0}\ \theta x_{1,0} \cdots\cdots \theta^{n-2} x_{1,0}) = F(x_{1,0})$$

where $F$ is a rational function. On replacing here $x_{1,0}$ by

$$x_{1,0}\quad x_{0,1}\quad x_{1,1}\ \cdots\cdots\ x_{n-1,1}$$

$t_\infty$ goes over into quantities which we denote respectively by

$$t_\infty,\ t_0,\ t_1 \cdots t_{n-1}$$

These are roots of an equation of degree $n+1$

$$W(t, x) = 0$$

whose coefficients lie in $R(x)$. In fact if

$$\iota \equiv \frac{\lambda}{\mu} \mod n$$

$$t_\iota = F(x_{\lambda,\mu}) = F(\theta x_{\lambda,\mu}) = \cdots = F(\theta^{n-2} x_{\lambda,\mu})$$

$$\iota = \infty, 0, 1 \cdots n-1$$

Hence

$$t_\iota^r = \frac{1}{n+1} \sum_h F^r(\theta^h x_{\lambda,\mu}) \qquad h = 0, \cdots n-2.$$

and thus

$$\sum_\iota t_\iota^r$$

is a symmetric function of the roots of $P(x, x) = 0$ and hence lies in $R(x)$; hence etc.

The equations $W = 0$ are Weber's equations of transformation. That our $T$ equation belongs to this class is seen thus. We observed at the commencement of this paragraph that

$$\operatorname{cn} w_{\lambda,\mu},\ \operatorname{dn} w_{\lambda,\mu}$$

were rational in $x^2_{\lambda,\mu}$; hence

$$y_\infty = \prod_{p=1}^{n} \frac{\operatorname{cn}}{\operatorname{dn}}(p w_{1,0})$$

is a rational symmetric function of

$$x^2_{1,0}\quad x^2_{2,0}\ \cdots\ x^2_{n,0}$$

and hence of

$$x_{1,0}\quad x_{2,0}\ \cdots\cdots\ x_{n-1,0}$$

that is of the roots in the first row of A.

2.

Having shown that the $n+1$ quantities

$$(1)\quad y_\infty = \Pi \frac{\mathrm{cn}}{\mathrm{dn}}\left(p\,\frac{4K}{n}\right),\quad y_a = \Pi \frac{\mathrm{cn}}{\mathrm{dn}}\left(p\,\frac{16a4K+4iK'}{n}\right)$$

$$p = 1, 2 \cdots m$$

$$a = 0, 1 \cdots n-1$$

are roots of a rational equation

$$(2)\quad T(y,x) = f_0(x)\,y^{n+1} + f_1(x)\,y^n + \cdots + f_{n+1}(x) = 0$$

we turn to the calculation of its coefficients.

To arrive at our goal it is necessary to know

($\alpha$) a superior limit of $x$ in $T$

($\beta$) how to develop the roots of $T$ into $q$-series.

Having these facts, the coefficients may be obtained in either of the two ways indicated by Sohnke.*

Let us consider ($\beta$) first. Setting as usual

$$\sqrt{k} = \frac{\vartheta_2(0\,|\,\tau)}{\vartheta_3(0\,|\,\tau)} = \sqrt{k(\tau)}$$

the transformation theory of the $\vartheta$'s gives us

$$\sqrt{k(n\tau)} = (\sqrt{k})^n \prod_{p=1}^{m} \frac{\mathrm{cn}^2}{\mathrm{dn}^2}\left(p\,\frac{4K}{n}\right) = (\sqrt{k})^n y'_\infty$$

$$\sqrt{k\left(\frac{\tau+16a}{n}\right)} = (\sqrt{k})^n \prod_{p=1}^{m} \frac{\mathrm{cn}^2}{\mathrm{dn}^2}\left(p\,\frac{16a4K+4iK'}{n}\right) =$$

$$(\sqrt{k})^n y_a^2$$

Hence extracting the square roots of these equations and setting for simplicity

$$(3)\qquad u = \varphi(\tau) = \sqrt{2}\; q^{\frac{1}{8}} \prod_{\nu=1}^{\infty} \frac{1+q^{2\nu}}{1+q^{2\nu-1}}$$

we have

$$y_\infty = \pm \frac{\varphi(n\tau)}{\varphi^n(\tau)}$$

$$y_a = \pm \frac{\varphi(\tau+16a)}{\varphi^n(\tau)}$$

* *Crelle*, vol. 16, p. 97, Aequationes modulares pro transformatione functionum ellipticarum.

The sign of $y_\infty$ we get by observing that if $\tau = r + is$

$$\lim_{s=\infty} \frac{\vartheta_2(v)}{\sqrt{k}} = 2\cos \pi v, \quad \lim_{s=\infty} \vartheta_3(v) = 1$$

and we have

$$\lim_{s=\infty} y_\infty = \lim \Pi \frac{1}{\sqrt{k}} \frac{\vartheta_2\left(\frac{2p}{n}\right)}{\vartheta_3\left(\frac{2p}{n}\right)} = \Pi \cos \frac{2\pi p}{n} = \frac{\gamma}{2^m}$$

where $$\gamma = \left(\frac{2}{n}\right) = (-1)^{\frac{n^2-1}{8}}$$

Hence $$y_\infty = \gamma \frac{\varphi(n\tau)}{\varphi^n(\tau)}$$

The sign of $y_0$ is at once obtained by observing that for $\tau = is$, $\vartheta_2(v)$, $\vartheta_3(v)$ are real and positive when $v$ is purely imaginary.

Thus $$y_0 = \frac{\varphi\left(\frac{\tau}{n}\right)}{\varphi^n(\tau)}.$$

The sign of the other roots we get by using the identity

$$(4) \qquad T[y_0(\tau), x(\tau)] = T\left[\frac{\varphi\left(\frac{\tau}{n}\right)}{\varphi^n(\tau)}, x(\tau)\right] = 0$$

If we replace here

$$\tau \text{ by } \tau + 16a$$

equation (3) shows that

$$\varphi(\tau + 16a) = e^{\frac{\pi i}{6} \cdot 16a} \varphi(\tau) = \varphi(\tau)$$

whence (4) becomes

$$T\left[\frac{\varphi\left(\frac{\tau + 16a}{n}\right)}{\varphi^n(\tau)}, x\right] = T(y_a, x) = 0$$

which shows that

$$y_a = \frac{\varphi\left(\frac{\tau + 16a}{n}\right)}{\varphi^n(\tau)}$$

is a root. As these are all distinct we have now our roots expressed as infinite $q$-products which may he converted into $q$-series either by multiplying and dividing out or by making use of the transformation theory of the $\vartheta$'s.* We find

$$y_\infty = \gamma 2^{-\frac{n}{3}}[1 + a_1 q + a_2 q^2 + \cdots]$$

$$(5) \qquad y_s = 2^{-\frac{n}{3}} q^{-\frac{r}{n}} \varepsilon^s [1 + \varepsilon^{2s} b_1 q^{\frac{1}{n}} + \varepsilon^{4s} b_2 q^{\frac{2}{n}} + \cdots]$$

$$\varepsilon = e^{\frac{2\pi i}{n}} \qquad r = \frac{rn^2 - 1}{8}$$

3.

We have now three different expressions for the roots of our $T$ equation from which we can at once draw some simple conclusions regarding the coefficients $f(x)$ of (2). In the first place the coefficient $f_0(x)$ must reduce to a monomial, since roots of

$$f_0(x) = 0$$

are the infinities of the $y$'s and these, as we now can see, are infinite only at $x = 0$. We can thus write $T = 0$ in the form

$$y^{n+1} + g_1(x) y^n + g_2(x) y^{n-1} + \cdots + g_{n+1}(x) = 0$$

where

$$g_s(x) = \frac{a_{s,0} + a_{s,1} x + \cdots + a_{s,m_s} x^{m_s}}{x^{r_s}}$$

The developments (5) in connection with Newton's formulæ

$$g_1 + s_1 = 0$$
$$2g_2 + s_1 g_1 + s_2 = 0$$
$$3g_3 + 3_1 g_2 + s_2 g_1 + s_3 = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

give us superior limits of the $r$'s. Consider first $r_{n+1}$: Evidently

$$\lim_{x=0} x^{r_{n+1}} (y_\infty \; y_0 \cdots y_{n-1}) = a_{n+1,0}$$

and (5) shows that this requires that

$$r_{n+1} = r \qquad a_{n+1,0} = \gamma$$

Consider now $r_1$; we have

$$-s_1 = g_1$$

*Sohnke *loc. cit.* gives $\varphi(\tau)$ to the 26th power of $q$; also the first 20 powers of $\varphi$ are given.

hence the order of $g_1$ for $x = 0$ is that of $s_1$, or since $y_\infty$ is finite that of

$$y_0 + y_1 + y_2 + \cdot + y_{n-1}$$

But (5) shows that this sum is

$$q^{-\frac{r}{n}}\left[b_0 \Sigma \varepsilon^h + b_1 q^{\frac{1}{n}} \Sigma \varepsilon^{h(h+1)} + b_2 q^{\frac{2}{n}} \Sigma \varepsilon^{h(2h+1)} + \cdots\right]$$

$$h = 0, 1 \cdots n-1$$

As $\sum_h \varepsilon^{h\rho}$ is $n$ or 0 according as $\rho$ is or is not $\equiv 0$ mod. $n$, we see that if $\nu_1$ is the smallest positive root of

$$8\nu + 1 \equiv 0 \text{ mod. } n$$

the order of $s_1$ is

$$\sigma_1 = \frac{r - \nu_1}{n}$$

In regard to the other exponents this method affords only a superior limit.* The orders

$$\sigma_2 \quad \sigma_3 \cdots$$

of

$$s_2 \quad s_3 \cdots$$

are found from (5) to be

$$\sigma_2 = \frac{2r - \nu}{n} \quad \text{where} \quad 4\nu_2 + 1 \equiv 0$$

$$\sigma_3 = \frac{3r - \nu_3}{n} \quad \cdots \quad 8\nu_3 + 3 \equiv 0$$

$$\sigma_4 = \frac{4r - \nu_4}{n} \quad \cdots \quad 2\nu_4 + 1 \equiv 0 \qquad \text{mod. } n$$

$$\sigma_5 = \frac{5r - \nu_5}{n} \quad \cdots \quad 8\nu_5 + 5 \equiv 0$$

.....

Where here $\nu$ is the smallest positive root of its corresponding congruence.

Newton's formulæ show us that $r_2$ is not superior to the greater of the orders of $s_2, s_1 g_1$ etc. In certain cases of frequent occurrence we need consider only the $\sigma$'s. Thus suppose we had found that the orders of

$$g_1 \quad g_2 \cdots g_n$$

* In § 8 we shall see that $r_n$ is also determined viz: $r_n = r$.

were not greater respectively than

$$\sigma_1 \quad \sigma_2 \cdots \sigma_a$$

then the order of $g_{a+1}$ is not superior to $\sigma_{a+1}$ if

$$\sigma_1 + \sigma_a,\ \sigma_2 + \sigma_{a-1},\ \sigma_3 + \sigma_{a-2}, \cdots$$

are not greater than $\sigma_{a+1}$. When the $\sigma$'s have this property we say they form a regular sequence.

4.

Another important and very evident property of the $T$ equation we get from the expression of the roots as quotients of the elliptic functions as given by (1). In fact, for $x = 1$ we know the elliptic functions degenerate to exponentials and

$$\lim_{x=0} \frac{\mathrm{cn}(u, x)}{\mathrm{dn}(u, x)} = 1 \tag{6}$$

if $u$ is independent of $x$. With us the case is somewhat different; here

$$u = p\,\frac{4\lambda K + 4\mu i K'}{n},$$

is not independent of $x$ and (6) does not always approach 1 as a limit for then $a_{n+1,0}$ could not be $\gamma = (-1)^r$ as we just found but would always be $+1$. It is easy to show* on passing to exponentials that

$$\lim_{x=1} \frac{\mathrm{cn}}{\mathrm{dn}}\left(p\,\frac{4\lambda K + 4\mu i K'}{n}\right) = \pm 1$$

according as

$$n - 4\lambda p \gtrless 0$$

Thus

$$T(y, 1) = (y - 1)\left(y - (-1)^r\right)^n$$

5.

We turn now to problem ($\alpha$) viz., determine a superior limit† of $x$ in $T$. By making use of the property that $T = 0$ remains

*Cf. A. ENNEPER: Elliptische Functionen, 2d Ed. 1890, p. 460.

†We remark that a superior limit of $x$ is given at once, for if

$$f(w^n, z^m) = 0$$

is irreducible, $m$ is equal to the sum of the orders of the infinities for the whole $z$ plane. For $T = 0$ this is precisely $r$.

unaltered for certain substitutions we can go much farther and write down the literal part of $T$ pretty closely. To do this we shall make use of the fact that the $W$ equations are irreducible in $R(x)$. From the Galoisian theory this is self-evident; but we can show it here by making use of a principle Weber* has employed in another connection.

In fact let

$$V(t, x)$$

be the irreducible factor of $W(t, x)$ admitting $t_\omega$ as root. In the identity

$$V[F(x_{1,0}(\tau);\ x(\tau)] = 0 \tag{7}$$

replace $\tau$ by

$$\tau_1 = \frac{c - a\tau}{b\tau - \partial} \tag{8}$$

where

$$a \equiv \partial \equiv 1 \quad b \equiv c \equiv 0 \mod 4 \quad a\partial - bc = 1 \tag{9}$$

Such a substitution leaves the coefficients of $V = 0$ unaltered. Let us see how it affects $t_\omega$.

The theory of linear transformation of the $\vartheta$'s gives

$$\frac{\vartheta_3(0 \mid \tau_1)}{\vartheta_2(0 \mid \tau_1)} \frac{\vartheta_1(v \mid \tau_1)}{\vartheta_0(v \mid \tau_1)} = e^{\frac{2\pi i}{4}(1-a)} \frac{\vartheta_3}{\vartheta_2} \cdot \frac{\vartheta_1}{\vartheta_0}\left(v(\partial - b\tau) \mid \tau\right) \tag{10}$$

$$a \equiv \partial \equiv 1 \quad b \equiv c \equiv 0 \mod 2 \quad a\partial - bc = 1$$

$$\tau_1 = \frac{a\tau - c}{\partial - b\tau} \tag{11}$$

But if we restrict the $a, b, c, \partial$ of (11) to satisfy also (9), the exponential factor drops out of (10). If we now set in (10)

$$v = \frac{2}{n}$$

it gives

$$x_{1,0}(\tau_1) = x_{b,-a}$$

The identity (7) after the substitution (8) becomes thus

$$V\left[F(x_{b,-a}),\ x\right] = 0$$

or setting

$$a\partial + b \equiv 0 \mod n$$

$$V(t_\omega, x)\, 0,$$

* Elliptische Functionen und Algebraische Zahlen, p. 221.

that is $V(t, x)$ admits all the roots of $W = 0$ and $W$ is thus irreducible.*

6.

We can now show at once that $T = 0$ is unaltered for the substitution

$$\begin{pmatrix} y & , & x \\ y^{-1}, & & x^{-1} \end{pmatrix} \tag{12}$$

For if in the identity

$$T[y_n(\tau)\,;\ x(\tau)] = 0$$

we replace $\tau$ by $\dfrac{\tau}{1-\tau}$ it goes over into

$$T\left[y_n\left(\frac{\tau}{1-\tau}\right), x^{-1}\right] = 0 \tag{13}$$

But the theory of the linear transformation of the $\vartheta$'s gives

$$\begin{aligned} y_n\left(\frac{\tau}{1-\tau}\right) &= \left(\sqrt{k\left(\frac{\tau}{1-\tau}\right)}\right)^{-n} \Pi \frac{\vartheta_2}{\vartheta_3}\left(p\cdot\frac{2}{n}\,\middle|\,\frac{\tau}{1-\tau}\right) \\ &= (\sqrt{k})^{n} \Pi \frac{\vartheta_3}{\vartheta_2}\left(\frac{2p(1-\tau)}{n}\right) \qquad p = 1, 2 \cdots m \\ &= \Pi \frac{\mathrm{dn}}{\mathrm{cn}}\left(p\cdot\frac{4K - 4iK'}{n}\right) \\ &= y_a^{-1} \end{aligned}$$

if $\qquad 16a + 1 \equiv 0 \mod n.$

Hence (13) becomes

$$T\left[y_a^{-1}, x^{-1}\right] = 0$$

that is the equation

$$T\left(y^{-1}, x^{-1}\right) = 0$$

of degree $n + 1$ has a root in common with the irreducible equation

$$T(y, x) = 0$$

and is thus identical with it.

* It is to be expressly remarked that this reasoning applies equally well to the domain formed by $\kappa$ and *any constants* $\alpha, \beta, \gamma \ldots$ independent of $\tau$.

Let us see how this helps us to determine the literal part of $T=0$. Apply (12) to $(2^e)$ and make in the resulting equation the coefficient of $y^{n+1}$ unity. The coefficients of the various powers of $y$ in the two equations must be identical. This shows at once that the absolute term must reduce to a monomial and thus its form is definitely

$$g_{n+1}(x)=\frac{\gamma}{x^r}$$

Let us look at the term

$$a_{\lambda,\mu}\frac{y^\lambda}{x^\mu}$$

In the new equation it becomes

$$\gamma a_{\lambda,\mu}\frac{y^{n+1-\lambda}}{x^{r-\mu}}$$

As no denominator in $g(x)$ can be of degree greater than $r$ we have

$$\mu \equiv\!\!\!\equiv 0$$

Also

$$a_{n+1-\lambda,\,r-\mu}=\gamma a_{\lambda,\mu}$$

$$\mu < r_{n+1-\lambda}$$

$$r-\mu < r_\lambda$$

Hence setting

$$\rho_\lambda = r - r_\lambda$$

we see the exponents $\mu$ lie within the narrow interval defined by

$$\rho_\lambda \equiv\!\!\!\equiv \mu \equiv\!\!\!\equiv r_{n+1-\lambda}$$

For example for $n=11$ we find by § 3

| $\lambda$ | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho_\lambda$ | 0 | 2 | 3 | 5 | 6 | 7 | 9 | 10 | 11 | 13 | 14 |
| $r_{n+1-\lambda}$ | 1 | 2 | 4 | 5 | 6 | 8 | 9 | 10 | 12 | 13 | 15 |

The $T$ equation thus has the form for $n=11$,

$$(14)\quad y^{12}+\frac{a+bx}{x}y^{11}+\frac{c}{x^2}y^{10}+\frac{d+ex}{x^4}y^9+\frac{f}{x^5}y^8+\frac{g}{x^6}y^7+$$

$$\frac{h+jx}{x^8}y^6-\frac{g}{x^9}y^5-\frac{f}{x^{10}}y^4-\frac{e+dx}{x^{12}}y^3-\frac{cy^2}{x^{13}}$$

$$-\frac{b+ax}{x^{15}}y-\frac{1}{x^{15}}=0$$

7.

Still other relations between the coefficients we get on showing that $T=0$ remains unaltered for the substitution

$$(15) \qquad \begin{pmatrix} y & x \\ \gamma y^{-\alpha} x^{-\tau} & x^{n} y^{\beta} \end{pmatrix}$$

In fact in the identity

$$T\left[y_{\omega}(\tau), x(\tau)\right] = 0$$

replace $\tau$ by $\frac{\tau}{n}$; it goes over into

$$T\left[\gamma y_0^{-\alpha} x^{-\tau}, \quad x^{n} y_0^{\beta}\right] = 0$$

Hence etc., as before. This shows that on applying (15) to $(2^{\alpha})$

$$a_{\lambda,\mu} \frac{y^{\lambda}}{x^{\mu}}$$

goes over into

$$\gamma^{\lambda+1} a_{\lambda,\mu} \frac{y^{\lambda'}}{x^{\mu'}}$$

in the new equation, where

$$\lambda' = n(n+1-\lambda) - 8\mu$$

$$\mu' = n\mu - (n-\lambda)\tau$$

and hence the coefficient of $\frac{y^{\lambda'}}{x^{\mu'}}$ is

$$a_{\lambda',\mu'} = \gamma^{\lambda+1} a_{\lambda,\mu}$$

For example for $n=11$ this shows that

$$-\partial = a \quad j = -c \quad -c = -h$$

This permits us to write (14) in the more precise form

$$(16) \quad y^{12} + \frac{(a+bx)}{x} y^{11} + \frac{cy^{10}}{x^2} + \frac{-a+ex}{x^4} y^9 + \frac{f}{x^5} y^8 + \frac{gy^7}{x^6} + \frac{c-cx}{x^8} y^6 - \frac{gy^5}{x^9} - \frac{fy^4}{x^{10}} + \frac{-e+ax}{x^{12}} y^3 - \frac{cy^2}{x^{13}} - \frac{b+ax}{x^{15}} y - \frac{1}{x^{15}} = 0$$

As here

$$T(y,1) = (y-1)\,(y+1)^{11} = 0$$

we see that

$$a + b = 10, \quad c = 44, \quad -a + e = 110$$
$$f = 165 \quad g = 132$$

Thus to get all the coefficients of (16) it is necessary to find one only of the three quantities $a$, $b$, $e$.

8.

This we get by remarking another property of our $T$ equation. In fact we saw § 2 that for $x = 0$, $y_\infty = \gamma 2^{-n}$ while all the other roots were infinite. If then we replace in $T = 0$, $y$ by $y^{-1}$ all the roots of the resulting equation will be zero except one which will be $\gamma 2^n$. This gives at once

$$r_n = r \qquad a_{n,0} = -2^n$$

For $n = 11$ this gives $a_{n,0} = -b = -32$
hence, $a = -22 \quad b = 32 \quad e = 88$ and our equation assumes its final form

$$(17)\quad y^{12} + \frac{-22 + 32x}{x} y^{11} + \frac{44}{x^2} y^{10} + \frac{22 + 88x}{x^4} y^9 + \frac{165}{x^5} y^8 +$$
$$\frac{132}{x^6} y^7 + \frac{44(1-x)}{x^6} y^6 - \frac{132}{x^9} y^5 - \frac{165}{x^{10}} y^4 - \frac{88 + 22x}{x^{12}} y^3 -$$
$$\frac{44}{x^{13}} y^2 - \frac{32 - 22x}{x^{15}} y - \frac{1}{x^{15}} = 0$$

For primes $> 11$ we must have recourse to the method indicated by Sohnke and referred to §2, to obtain all the coefficients.

9

In conclusion let us look back and compare the $T$ equation with the usual modular equation. We observe that whereas the modular equation in the usual theory occupies a position apart requiring a separate treatment, the $T$ equation is only one of a great family of equations, the $W$ equations. If we deduct Arts. 1 and 5 which concern the rationality of the coefficients and the irreducibility and which pertain to all the equations of this class, we see the theory of our $T$ equation reduces to very small dimensions. Finally we pass from $T = 0$ to $M = 0$ on setting $y = u^{-n} v$. These three counts seem to make it desirable to replace the

historical modular equation by the $T$ equation which may be regarded as the modular equation suitably normed from our present standpoint.

If we desire an equation whose roots like those of the modular equations remain finite for finite $x$, we should replace the $T$ equation by the equation $U(z,x) = 0$ generated by

$$z_\omega = \Pi \frac{dn}{cn}\left(p\frac{4K}{n}\right) = y_\omega^{-1}. \qquad p = 1, 2 \cdots m.$$

For $n = 11$ such an equation is got from (17) by replacing $y$ by $y^{-1}$, we have then

$$z^{12} + (32 - 22x)\, y^{11} + 44 x^2 z^{10} + (88 + 22x)\, x^3 z^9 + 165 x^5 z^8$$
$$+132 x^6 z^7 - 44(1-x)\, x^7 z^6 - 132 x^9 z^5 - 165 x^{10} z^4 - (22 + 88x) x^{11} z^3$$
$$- 44 x^{13} z^2 + (22 - 32x)\, x^{14} z - x^{15} = 0.$$

The theory of these equations can, of course, be made independent of the $T$ equations.

NEW HAVEN, CONN.,
*March*, 1897.

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## CORRECTION.

The following errata occur in the abstract of Professor Felix Klein's Princeton Lecture "On the Stability of a Sleeping Top," printed in the January number of the BULLETIN, pp. 129–132:

At the bottom of page 130, read

$$U = 2(u-1)\,(n^2 + (Pu - h)\,(u+1)).$$

In the middle of p. 131, read

$$v^2 = \frac{(1-e)\,(n^2 - 2P(e+1))}{e+1}.$$

On page 132, fifth line from the bottom, read $n^2 - 4P$ instead of $u^2 - 4P$.

## NOTES.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York on Saturday afternoon, April 24, at three o'clock, the Vice-President, Professor R. S. Woodward, in the chair. There were five members present. The Council announced the election of the following persons to membership in the Society: Mr. CHANCY W. BOUCHER, Marion Normal College, Marion, Ind.; Dr. LINNEAUS WAYLAND DOWLING, University of Wisconsin, Madison, Wis.; Mr. SOLOMON A. JOFFE, Columbia University, New York, N. Y. Three nominations for membership were received. The following papers were read:

(1) Dr. E. O. LOVETT: "Note on the invariants of $n$ points."

(2) Professor H. B. NEWSON: "Continuous groups of circular transformations."

(3) Dr. L. E. DICKSON: "Systems of continuous and discontinuous simple groups."

AT the meeting of the Council held April 24, it was proposed that the New York meetings of the Society be held hereafter at intervals of two months. By providing for both morning and afternoon sessions at each meeting as much time would be allowed for the business of the Society as at present. Action upon this proposition was deferred until the next meeting of the Council.

THE UNIVERSITY OF CHICAGO, SUMMER 1897. The following mathematical courses will be offered:—By Professor MOORE: Abstract groups; Projective geometry.—By Professor BOLZA: Hyperelliptic functions; Advanced integral calculus.—By Dr. LOVETT: The geometry of Lie's transformation-groups.—By Dr. YOUNG: [1]Conferences on mathematical pedagogy; [1]Determinants; Culture calculus; [2]Plane trigonometry.—By Mr. SLAUGHT: Integral calculus; College algebra. The courses are four or five hours weekly for twelve weeks from July 1, 1897; the two courses marked 1 are, however, only for the first six weeks, and the course marked 2 is ten hours weekly for the second six weeks. Those who expect to work in mathematics at the University of Chicago during the coming summer, as well as those who desire further information, are requested to communicate with Professor Moore.

UNIVERSITY OF GÖTTINGEN. The mathematical courses announced for the summer semester are the following:—By Professor SCHERING: Partial differential equations; Seminar in mathematical physics.—By Professor VOIGT: Potential; Selected chapters of mechanics in the seminar.—By Professor KLEIN: Introduction into the theory of differential equations; Seminar in mathematical physics.—By Professor SCHUR: Theoretical astronomy (second part) and theory of perturbations; Seminar in mathematical physics.—By Professor HILBERT: Algebraic theory of invariants with applications to geometry; Theory of functions (second part); Exercises in theory of functions in seminar (with Professor KLEIN).—By Professor SCHOENFLIES: Determinants; Non-Euclidean geometry; Mathematical proseminar.—By Dr. BOHLMANN: Analytical geometry; Mathematical exercises in insurance.—By Dr. SOMMERFELD: Differential calculus with exercises; Calculus of variations.

UNIVERSITY OF LEIPZIG. The mathematical courses announced for the summer semester are the following:—By Professor SCHEIBNER: On the variation of constants in problems of mechanics.—By Professor NEUMANN: Selected chapters of the theory of functions; Discussion of mathematical problems.—By Professor LIE: Introduction into analytical geometry; Differential equations with infinitesimal transformations; Exercises in the mathematical seminar.—By Professor MAYER: Higher analytical mechanics; Exercises in the mathematical seminar.—By Professor VON OETTINGEN: Meteorology.—By Professor ENGEL: Applications of differential calculus to curves and surfaces; Mathematical seminar.—By Professor DRUDE: Theory of heat.—By Dr. WIEDEBURG: Electrical oscillations.—By Dr. HAUSDORFF: Applications of differential calculus to natural science.—By Dr. KNOBLAUCH: Introduction to the mathematical treatment of questions in natural science.

UNIVERSITY OF MUNICH. The mathematical courses announced for the summer semester are the following:—By Professor BAUER: Algebra (theory of equations), continuation; Mathematical seminar.—By Professor LINDEMANN: Analytical geometry of three dimensions; Theory of definite integrals and analytical representation of arbitrary functions; Mathematical seminar.—By Professor PRINGSHEIM: Integral calculus; Exercises; Applications of differential calculus to geometry.—By Professor GRÄTZ: Mechanical theory of heat.—By Dr. BRUNN: Elements of higher mathematics for students of all departments.—By

Dr. DÖHLEMANN: Descriptive geometry (second part); Exercises; Graphical statics; Exercises.—By Dr. VON WEBER: Selected chapters of algebraic analysis; Elementary geometry; Determinants with applications.—By Dr. KORN: Theory of potential; D'Alembert and Lagrange.

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### I. HIGHER MATHEMATICS.

BARKER (A. H.). Graphical calculus: with an introduction by J. Goodman. London, Longmans, 1896. 12mo. 10 and 188 pp. Cloth.

BOBEK (K.). Einleitung in die projektivische Geometrie der Ebene. Ein Lehrbuch für höhere Lehranstalten und für den Selbstunterricht. Nach den Vorträgen des Herrn C. Küpper bearbeitet. 2te wohlfeile Ausgabe. Leipzig, Teubner, 1897. 8vo. 6 and 210 pp. Mk. 2.00

CABREIRA (A.). Sur la géométrie des courbes transcendantes. Mémoires originaux. Lisbonne, 1896. 8vo.

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FIEGER (A.). Ueber die hypergeometrische Reihe und die Zurückführung anderer Reihen auf dieselbe. Karlsbad, 1896. 8vo. 20 pp. Mk. 1.50

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GLAS (R.). Krümmungshalbmesser-Constructionen der Kegelschnitte. Wien, 1896. 8vo. 21 pp. Mk. 1.50

GMEINER (J. A.). Ueber die ganzen Zahlen im Rationalitätsgebiete der fünften Einheitswurzeln. Pola, 1896. 8vo. 24 pp. Mk. 1.50

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KIEPERT (L.). Tabelle der wichtigsten Formeln aus der Integral-Rechnung. 6te Auflage. Hannover, Helwing, 1897. 8vo. 38 pp. Mk. 0.50

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RUDZKI (T.). Ueber Ampère's und Darboux's Integrationstheorie der partiellen Differentialgleichungen 2ter Ordnung. Leipzig, 1896. 8vo. 51 pp. Mk. 1.80

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## II. ELEMENTARY MATHEMATICS.

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